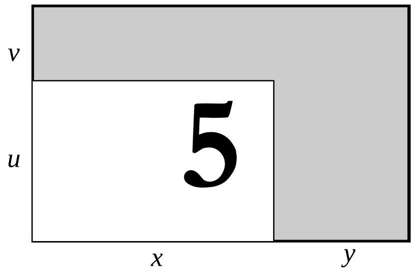


5 Algebra

A multiplication trick

What is 101 × 26? And what about 101 × 47? Find some more products of 101 by two-digit numbers and check. Why do we get the same digits repeated?

We can write 101 × 23 as (100 + 1) × 23, so that

$$\begin{aligned} 101 \times 23 &= (100 + 1)\ 23 \\ &= 2300 + 23 = 2323 \end{aligned}$$

What general principle are we using here?

Like this, which number multiplied by three-digit numbers make the digits repeat? (Look at the section *Dividing again and again*, in the lesson *Number World* of the Class 5 textbook)

Product of sums

The general principle we used in the problems above is that in order to multiply the sum of two numbers by a third number, we have to multiply each number in the sum separately and add.

Using algebra,

$$(x + y)\ z = xz + yz \text{ for all numbers } x, y, z$$

Suppose we want to multiply the sum of two numbers by another sum of two numbers?

For example, how do we find (8 + 6) × (10 + 5)?

There are several methods, right?

We can write

$$(8 + 6)\ (10 + 5) = 14 \times 15 = 210$$

Likewise,

$$(8 + 6)\ (10 + 5) = 14 \times (10 + 5)$$

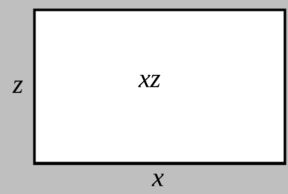
Algebra and Geometry

We can give a geometric explanation of the fact that

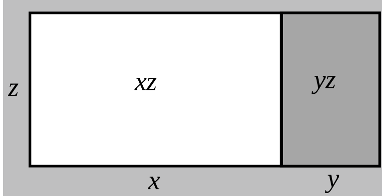
$$(x + y)\ z = xz + yz$$

in the case when x, y, z are positive numbers.

The area of the rectangle with lengths of sides x and z is xz, isn't it?



Suppose we extend the side of length x and enlarge the rectangle? If the increase in length is y, then the lengths of the sides of this newer and larger rectangle are $x + y$ and z.



So, the area of the new rectangle is $(x + y)z$.

We can see in the picture that this new rectangle is made up of the original rectangle and another rectangle put together. Their areas are xz and yz and the area of the large rectangle is the sum of these areas.

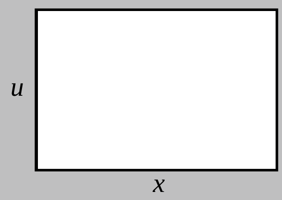
Thus

$$(x + y)\ z = xz + yz$$

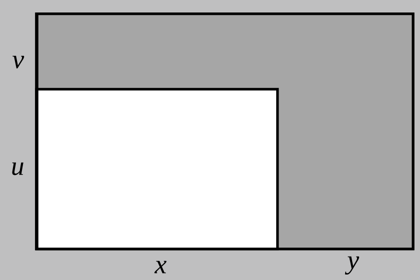
Geometry again

The products of sums of positive numbers can also be explained through geometry.

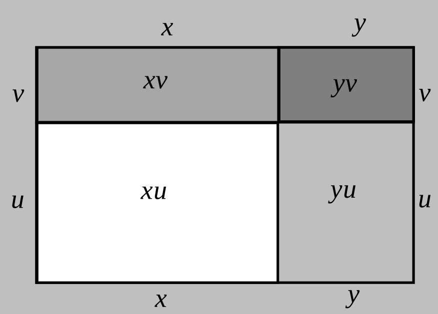
We start with the numbers x, y, u, v. First we draw a rectangle with the lengths of sides x and u.



Next we enlarge the rectangle, by extending the sides, so that one side is of length $x + y$ and the other is of length $u + v$.



This larger rectangle can be divided into four pieces as shown below:



The area of the large rectangle is the sum of the areas of these four small pieces and this gives us

$$(x + y)\ (u + v) = xu + xv + yu + yv$$

$$= (14 \times 10) + (14 \times 5)$$

$$= 140 + 70$$

$$= 210$$

(This is what we actually do, when we find the product 14×15 in the "usual" way of writing the products one below the other and adding.) Another method of finding this product is the following

$$(8 + 6)\ (10 + 5) = (8 + 6) \times 15$$

$$= (8 \times 15) + (6 \times 15)$$

$$= 120 + 90$$

$$= 210$$

Let's look at the general algebraic method of finding such a product. Suppose we want to find out $(x + y)\ (u + v)$.

For convenience, let's denote the sum $u + v$ by z. Then

$$(x + y)\ (u + v) = (x + y)\ z$$

We also know that,

$$(x + y)\ z = xz + yz$$

Now let's use the fact that $z = u + v$. Then

$$x\ z = x\ (u + v) = xu + xv$$

$$y\ z = y\ (u + v) = yu + yv$$

What do we get when we put all these pieces together?

For all numbers x, y, u, v

$$(x + y)\ (u + v) = xu + xv + yu + yv$$

What does this mean?

To find the product of a sum of two numbers by another sum of two numbers, we have to multiply each number of the first sum by each number of the second sum and add all these products together.

So, we have another method of doing our first problem:

$$(8 + 6)\,(10 + 5) = (8 \times 10) + (8 \times 5) + (6 \times 10) + (6 \times 5)$$
$$= 80 + 40 + 60 + 30$$
$$= 210$$

How do we find $(2x + y) \times (3u + 2v)$, using our general principle?

$$(2x + y)\,(3u + 2v) = (2x \times 3u) + (2x \times 2v) + (y \times 3u) + (y \times 2v)$$
$$= 6xu + 4xv + 3yu + 2yv$$

Can't you now find these products on your own?

- $(p + q)\,(2m + 3n)$
- $(4x + 3y)\,(2a + 3b)$
- $(4a + 2b)\,(5c + 3d)$
- $(m + n)\,(5a + b)$
- $(2x + 3y)\,(x + 2y)$
- $(3a + 2b)\,(x + 2y)$

Product of differences

Whatever be the numbers $x, y, u, v,$ positive, negative or zero, we have

$$(x + y)\,(u + v) = xu + xv + yu + yv$$

So, this will be true even if we replace y by $-y$. That is,

$$(x + (-y))\,(u + v) = xu + xv + (-y)\,u + (-y)\,v$$

Here, what is the meaning of the $x + (-y)$ on the left side?

We have seen that

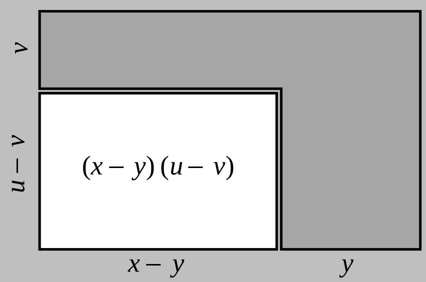
$$x + (-y) = x - y$$

in the lesson, *Negative Numbers*.

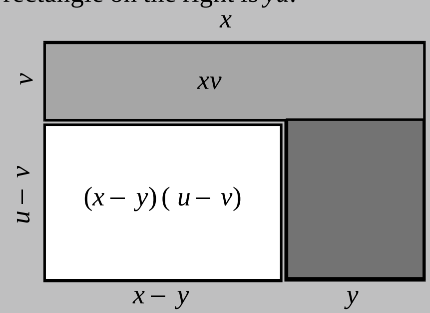
What about the meaning of $(-y)\,u$, and $(-y)\,v$ on the right?

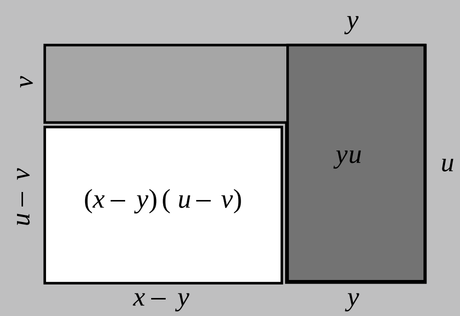
Product of differences in geometry

Starting with a rectangle with lengths of sides x and u, suppose we make it smaller by subtracting y from the first side and v from the second side, the area of the original rectangle is xu and the area of the smaller rectangle is $(x - y)\,(u - v)$

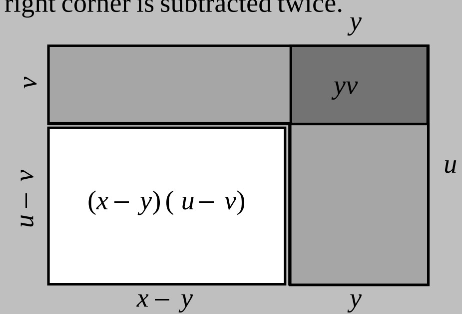


Look at the piece cut off. The area of the top rectangle is xv and the area of the rectangle on the right is yu.





If we subtract both these from the original area, the area of the rectangle on the top-right corner is subtracted twice.



To set this right, we have to add this area once. That is,

$$(x - y)\,(u - v) = xu - xv - yu + yv$$

Alternate multiplication

How do we compute 32 × 46?

$$\begin{array}{r} 32\ \times \\ 46 \\ \hline 192 \\ 1280 \\ \hline 1472 \end{array}$$

(Usually, we don't write the 0 in the second product.)

What is the principle used in this scheme?

$$\begin{aligned} 32 \times 46 &= 32 \times (6 + 40) \\ &= (32 \times 6) + (32 \times 40) \\ &= 192 + 1280 \\ &= 1472 \end{aligned}$$

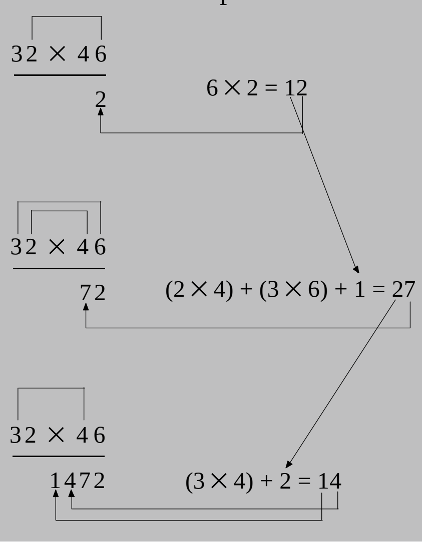
Instead, we can also use the computation below:

$$\begin{aligned} 32 \times 46 &= (2 + 30) \times (6 + 40) \\ &= (2 \times 6) + (2 \times 40) \\ &\quad + (30 \times 6) + (30 \times 40) \\ &= 12 + 80 + 180 + 1200 \\ &= 1472 \end{aligned}$$

As in the usual method, we can write the products one below another and add:

$$\begin{array}{r} 12 \\ 80 \\ 180 \\ 1200 \\ \hline 1472 \end{array}$$

The scheme below shows how this procedure can be simplified:



We have seen that

$$(-y)\,u = -yu \quad \text{and} \quad (-y)\,v = -yv$$

So how do we write the product we started with?

$$(x - y)\,(u + v) = xu + xv - yu - yv$$

Similarly, we can see that

$$(x + y)\,(u - v) = xu - xv + yu - yv$$

How about $(x - y)\,(u - v)$?

$$\begin{aligned} (x - y)\,(u - v) &= xu + x\,(-v) + (-y)\,u + (-y)\,(-v) \\ &= xu - xv - yu + yv \end{aligned}$$

Now find these products on your own:

- $(x + 3y)\,(2a - b)$
- $(3x + 5y)\,(3m - 2n)$
- $(2r - 3s)\,(t - u)$
- $(a - b)\,(4x - 3y)$
- $(3a - 5b)\,(2c - d)$
- $(2p + 5q)\,(3r - 4s)$

Square of a sum

Recall that the square of a product of two numbers is equal to the product of the squares of these numbers (see the **section,** *Square product*, of the lesson *Square Numbers* in the Class 7 textbook).

For example,

$$(5 \times 2)^2 = 10^2 = 100 = 25 \times 4 = 5^2 \times 2^2$$

Like this, is the square of a sum equal to the sum of squares?

We have $(5 + 2)^2 = 49$; but $5^2 + 2^2 = 29$

Let's see if there is any relation between the square of a sum and the sum of squares. The square of the sum of two numbers x and y is $(x + y)^2$, right?

How about writing this as the product $(x + y)\,(x + y)$ and then use our rule for the product of sums?

$$(x + y)^2 = (x + y)\,(x + y)$$

$$= (x \times x) + (x \times y) + (y \times x) + (y \times y)$$

Here

$$x \times x = x^2 \qquad y \times y = y^2$$

Also, we know that

$$xy = yx$$

So,

$$(x + y)^2 = x^2 + xy + xy + y^2 = x^2 + 2xy + y^2$$

So, what's the relation between the square of a sum and the sum of squares?

In algebraic language, we have the following:

For all numbers x and y

$(x + y)^2 = x^2 + y^2 + 2xy$

What is its meaning in ordinary language?

The square of the sum of two numbers is equal to the sum of their squares and twice their product.

Let's see how this helps us to compute 101.

$$101^2 = (100 + 1)^2$$

$$= 100^2 + 1^2 + 2 \times 100 \times 1$$

$$= 10000 + 1 + 200$$

$$= 10201$$

Now can you compute 201 similarly in your head?

In general, we have

$(x + 1)^2 = x^2 + 1 + 2x = x^2 + 2x + 1$ for any number x.

Squaring trick

We saw how our general principle about the product of sums can be used to get a simple method to compute the product of two-digit numbers. We now see how the general principle about the square of a sum can be similarly used to get a method to compute the squares of two-digit numbers.

For example, we have

$$46^2 = (6 + 40)^2$$

$$= 6^2 + (2 \times 6 \times 40) + 40^2$$

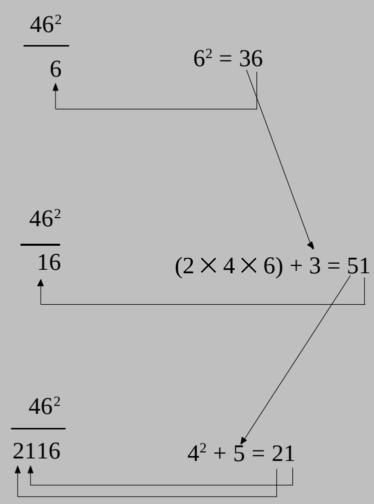
$$= 36 + 480 + 1600$$

$$= 2116$$

We can write the three products one below another and add:

$$\begin{array}{r} 36 \\ 480 \\ 1600 \\ \hline 2116 \end{array}$$

We can simplify the procedure as shown below:



Queer squares

What is 15^2? What about 25^2? For any number ending in 5, will the last two digits of the square be 25?

Why is this so?

Try $(10x + 5)^2$. Got it?

Let's see if we can find an easy procedure to compute the other digits in such squares.

$$①\ 5^2 = ②\ 25$$

$$②\ 5^2 = ⑥\ 25$$

$$③\ 5^2 = ⑫\ 25$$

Do you see any relation between the circled numbers on either side of each equation? Look at the expansion of $(10x + 5)^2$ once more:

$$\begin{aligned}(10x + 5)^2 &= 100x^2 + 100x + 25\\ &= 100x\,(x + 1) + 25\end{aligned}$$

Found the trick?

Now can't you do these problems?

- Find the squares of the following algebraic expressions.
  - $2x + 3y$
  - $x + 2$
  - $2x + 1$
- Compute the squares of these numbers mentally.
  - 102
  - 202
  - 1001
  - 2002
  - 205
  - 10.3
- Prove that $x^2 + 6x + 9$ is a perfect square for all natural numbers x. What can you say about its square root?
- Prove that in the sequence 1, 2, 3, . . . of natural numbers, 1 added to the product of any two alternate numbers is a perfect square.

Difference square

We have found an algebraic expression for the square of the sum of two numbers. Noe let's look at the square of a difference.

For any two numbers x and y, we have

$$x - y = x + (-y).$$

So,

$$\begin{aligned}(x - y)^2 &= \big(x + (-y)\big)^2\\ &= x^2 + (-y)^2 + 2x\,(-y)\end{aligned}$$

In this, we have

$$(-y)^2 = (-y) \times (-y) = y \times y = y^2$$

and

$$2x\,(-y) = 2\,(-xy) = -2xy$$

So, how do we write $(x - y)^2$?

For all numbers x and y,

$(x - y)^2 = x^2 + y^2 - 2xy$

How do we say this in ordinary language?

Let's see how we can use this to compute 99^2?

$$\begin{aligned} 99^2 &= (100 - 1)^2 \\ &= 10000 + 1 - 200 \\ &= 9801 \end{aligned}$$

Now's the time for some questions.

- Find the squares of the algebraic expressions given below:
  - $2x - 3y$
  - $x - 2$
  - $2x - 1$
- Mentally compute the squares of the numbers given below:
  - 98
  - 198
  - 999
  - 1998
  - 195
  - 9.7
- Prove that $x^2 - 6x + 9$ is a perfect square for all natural numbers x. What can you say about its square root?

Geometric algebra

We have mentioned the Master Geometer Euclid and his work *Elements* in the Class 7 textbook. The second part of Elements discusses some of the general principles we have seen in this chapter. The only thing is, they are given in the language of geometry rather than algebra.

For example, we have seen the general principle:

> The square of the sum of two numbers is equal to the sum of their squares and twice their product.

This can be written algebraically as

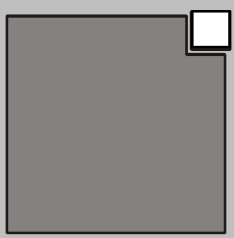
$$(x + y)^2 = x^2 + y^2 + 2xy$$

The same general principle, Euclid states thus:

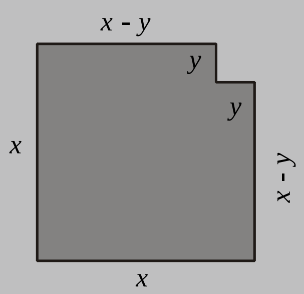
> The area of a square drawn on a line is equal to the sum of the areas of the squares on the two lines got by cutting this line at any point, and twice the area of the rectangle formed by these two lines.

Square difference

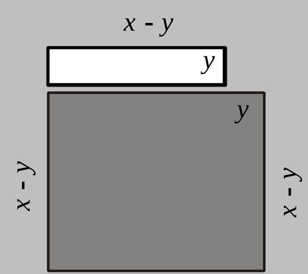
Suppose that from one corner of a square with length of sides x, a small square with length of the sides y is cut off:



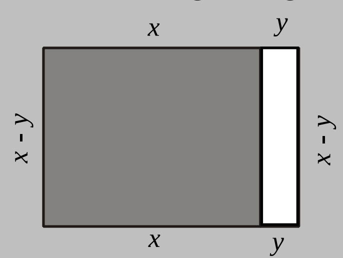
The area of remaining part is $x^2 - y^2$.



Cut out a rectangle of width y from the top of this figure:



And paste this rectangle onto the right side of the remaining rectangle:



The area of the rectangle so obtained is $(x + y)(x - y)$, isn't it?

Thus

$$x^2 - y^2 = (x + y)(x - y).$$

Sum and difference

We have seen general principles for the sum or difference of two numbers multiplied by itself (that is, squared). How about the product of the sum of two numbers by their difference?

$$\begin{aligned}(x + y)(x - y) &= (x + y)\big(x + (-y)\big)\\ &= (x \times x) + \big(x \times (-y)\big) + (y \times x) + \big(y \times (-y)\big)\\ &= x^2 - xy + yx - y^2\\ &= x^2 - xy + xy - y^2\\ &= x^2 - y^2\end{aligned}$$

Thus we find this general rule:

For all numbers x and y,

$(x + y)(x - y) = x^2 - y^2$

How do we say this in ordinary language?

> The product of the sum of two numbers by their difference is equal to the difference of their squares .

As an example, let's see how we can use this to find the product 45×35 easily:

$$\begin{aligned}45 \times 35 &= (40 + 5) \times (40 - 5)\\ &= 40^2 - 5^2\\ &= 1600 - 25\\ &= 1575\end{aligned}$$

Can't you easily find the products below like this?

- 51×49
- 98×102
- 10.2×9.8
- 7.3×6.7

Difference of squares

Let's write in reverse, our general principle on the product of sum and difference

$$x^2 - y^2 = (x + y)(x - y)$$

This means

> The difference of the squares of any two numbers is equal to the product of their sum and difference.

For example, suppose that the hypotenuse of a right angled triangle is 53 centimeters and the length of one short side is 28 centimeters. How do we compute the length of the third side?

By Pythagoras Theorem, the length of the third side is $\sqrt{53^2 - 28^2}$, isn't it? As we have seen just now,

$$53^2 - 28^2 = (53 + 28)(53 - 28) = 81 \times 25$$

So,

$$\sqrt{53^2 - 28^2} = \sqrt{81 \times 25} = \sqrt{81} \times \sqrt{25} = 9 \times 5 = 45$$

Thus the length of the third side is 45 centimeters.

Now try your hand at these problems:

- Compute the following in your head:
  - $67^2 - 33^2$
  - $123^2 - 122^2$
  - $\left(\frac{5}{7}\right)^2 - \left(\frac{2}{7}\right)^2$
  - $0.27^2 - 0.23^3$
- The diagonal of a rectangle is 65 centimeters long and one of its sides is 63 centimeters long. What is the length of the other side?

Pythagorean triplets

We have defined a Pythagorean triplet as three natural numbers such that the square of one number is the sum of the squares of the other two, in the Class 7 textbook.

For example,

$$3^2 + 4^2 = 5^2$$

and so the three numbers 3,4,5 form a Pythagorean triplet. A clay tablet from ancient Babylonia, dated around 2000 BC contains a list of such triplets.

There is a general method to find all such triplets. Let m, n be any two natural numbers and let the numbers x, y, z be computed as follows:

$$x = m^2 - n^2$$
$$y = 2mn$$
$$z = m^2 + n^2$$

Then we can easily seen that

$$x^2 + y^2 = z^2$$

Why does this work? Note that

$$\begin{aligned} x^2 + y^2 &= (m^2 - n^2)^2 + (2mn)^2 \\ &= m^4 + n^4 - 2m^2n^2 + 4m^2n^2 \\ &= m^4 + n^4 + 2m^2n^2 \\ &= (m^2 + n^2)^2 \\ &= z^2 \end{aligned}$$

This was known to Greek mathematicians as early as 300 BC.

- Prove that the difference of the squares of two consecutive natural numbers is equal to their sum.
- In how many different ways you can write 15 as the difference of the squares of two natural numbers?